Bow and Arrow in the Groves of Ind

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The Roman poet Vergil (70 BC - 19 BC) has some lines in his *Georgics* which run, in translation, as follows:

> "Or why should I mention the groves that India close to Ocean bears, a recess of the farthest circle of the world, where not a single arrow in its flight has been able to conquer the topmost loft of a tree?"

This is intriguing as a way to measure the height of a tree, and leads us to ask how high these trees were, and what sort of bow shot these arrows.

Let us begin with some very basic considerations. If we neglect air resistance, we may determine the trajectory of an arrow from a knowledge of two parameters. The are V, the velocity with which it is released, and α, the angle of discharge. We may follow its flight by taking co-ordinates x horizontally in the direction of the motion and y vertically up. See Figure 1. The equations of motion may now be easily written down from a knowledge of Newton's laws. We have:

$$\ddot{x} = 0 \text{ and } \ddot{y} = -g \tag{1}$$

where g is a constant known as the acceleration due to gravity and the double dots indicate second derivatives with respect to time. In SI units, g has the value 9.81, in other words about 10.

Equations (1) may be integrated to give two equations for the velocity components of the arrow. This gives:

$$\dot{x} = V\cos\alpha \text{ and } \dot{y} = V\sin\alpha - gt, \tag{2}$$

where t is the time elapsed since the arrow was released. We may now integrate these equations yet again to find

$$x = (V\cos\alpha)t \text{ and } y = (V\sin\alpha)t - \frac{1}{2}gt^2 \tag{3}$$

If we now combine the two equations (3), eliminating t, we reach

$$y = x\tan\alpha - (\frac{g}{2V^2}\sec^2\alpha)x^2. \tag{4}$$

This is the equation of the arrow's trajectory and it is readily seen to be parabolic.[1] It may be proved (and is fairly obvious in any case) that the maximum height will be attained when the arrow is fired vertically, i.e. α=90°. In that case, it will reach a height H given by

$$H = \frac{V^2}{2g}. \tag{5}$$

(Work this out as an exercise!)

When it comes to covering the greatest horizontal distance (the range), it turns out (again we leave this to the reader as an exercise) that we should choose α=45°, in which case we find a range R given by

$$R = \frac{V^2}{g}. \tag{6}$$

Comparing Equations (5) and (6), we find a simple result:

$$H = \frac{1}{2}R. \tag{7}$$

So, as long as air resistance is neglected, we have an easy key to the question we asked. Although H is somewhat difficult to measure, R is not. There is quite an amount of data on *how far* a longbow can shoot. The English longbow that was so deadly in medieval warfare was probably as efficient as a wooden "self-bow" can be. (A "self-bow" is a bow whose body is shaped out of a single homogeneous piece of material and is not laminated or reinforced.) These had ranges in the region 150-200m, and for ease of subsequent calculation, we will take 160m as a standard range.

So Vergil is saying (or is he?) that the trees in the groves of India are over 80m high. This seems wrong. Although the giant Sequoias of North America can attain such heights, mature rainforest trees typically stand only some 40-45m tall. Why then the discrepancy? Was Vergil indulging in poetic licence? Or were we wrong to neglect air resistance?

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[1] For more on parabolic trajectories, see *Function*, Vol 16, Part 4, p. 100.

Well, we'll come back later to the question of air resistance in the Appendix, but the main answer we would suggest to the question posed by the disparity is that Vergil was not referring to the *military* bow (which the Romans did not use), but to some other sort of bow. And if we posit this, we must ask what other kind of bow there was. Now there *were* bows in military use in Indian antiquity; indeed a passage in the *Mahâbhârata* (a classic Indian epic) has warriors shooting arrows very far into the sky. But we think it was not these military bows that Vergil had in mind.

Throughout India and Southeast Asia, one can still find descendants of the original t[2]ribes that lived there long before the invasion of the now-dominant cultural groups, and many of these tribes used the bow and arrow for purposes of hunting rather than of warfare. In many instances their bows were of a much lighter construction than the military bows of (e.g.) the English archers. They were not intended to be fired over great distances, nor high into the sky. Often the arrows were tipped with poison and they were fired from quite short range.

Indeed, strange as it may seem, our word *toxic*, meaning "poisonous", derives from an earlier word for "bow". It reached English via the Latin *toxicum*, meaning "a poison", but the Latin is in its turn derived from the Greek *toxikon pharmakon*, which meant "arrow poison". The word *pharmakon* was dropped (its modern day derivatives mean "pertaining to drugs", rather than specifically "poison") and the word *toxicon* retained. This derived from the Greek *toxon*, a bow, and this in its turn came from an Asiatic souce, the Scythian **taksha-*, also meaning "bow". Thus the word *toxic* itself refers to a practice that was followed in Asia but in hunting rather than in warfare.

It is also true that the Greeks saw the bow as of Asian origin and in this they were followed by the Romans. Thus, Vergil was most likely to be referring to the indigenous bow of India, and his lines then make quite literal sense. Stories of India would have reached him along with the spices that travelled the same route (and sold in Roman markets cheaply, and thus in quantity, as Vergil's contemporary Horace tells us), and here is the way in which Vergil would have learned of the size of the trees of India, as measured by the bow and arrow of that same land.

Appendix

It is now time to consider the question of air resistance. Equations (1) to (7) refer to an arrow fired in a vacuum, and this may not be accurate enough an approximation for our purposes. After all we would expect the path of the arrow to be affected by the breeze! However, let us suppose that the arrow flies through still air and and that *this* retards its motion. Precisely the same principle is involved. Now the interaction is in fact quite complex, but the best and most widely used approximation supposes that the air resistance supplies a force directed exactly against the travel and of a magnitude proportional to the square of the velocity.

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[2] The Scythians were Asain nomads who, between the 7th and 1st centuries BC, settled in what is now the Ukraine on the noorthern shores of the Black Sea. The asterisk indicates an inferred word (rather than one still preserved) for Scythian was never a written language.

This leads to two equations, which the reader should compare with Equations (1).

$$\ddot{x} = -K\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\dot{y} = -g - K\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2} \qquad (8)$$

where K is a constant still to be determined.[3]

The value of K depends on many factors;[4] here we shall note that we have already chosen g =10 and R=160 in SI units. This gives V≈40, again in SI units (here ms^{-1}). K depends on V and also on a number of other parameters: the density of air, the density of the the wood in the arrow, the length of the arrow, the so-called "kinematic viscosity" of air, and the approximate width of the arrow. We may estimate or look up these to find values in SI units of respectively 1.225x10^{-3}, 0.5, 1, 1.5x10 $^{-5}$, 0.01. The second, third and last of these figures are rough approximations; we have an arrow made of relatively light wood (about the middle of a large range) about 1m long and about a centimetre across. These figures do not affect the calculation very much; the most critical is the density and even that does not matter to any great extent.

Now an arrow is supposed to be a slender, streamlined body. It comprises a long thin shaft, a head and (at the tail end) a set of *fletchings* (often feathers or such) that act to keep the head pointing in the direction of travel. These last act by localising the drag of the air, so that while the arrow is correctly aligned air-resistance is minimised. The air-flow over the arrow is thus that for a well-streamlined body and tables are available. We find a value of K of about 3x10^{-4} in SI units.

Now 3x10^{-4} seems quite small, but we need to say "small compared to what?", and here somne further analysis is called for. The way to do this analysis is to adopt natural units in which V=1 and g=1. Up till now we have been using SI units, but these have no particular connection with the problem in hand.[5] We will do much better to choose especially apropriate units.

In these new units, the range is 1, H=0.5 and the equations become:

$$\dot{x} = -\varepsilon\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2}$$
$$\dot{y} = -1 - \varepsilon\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2} \qquad (9)$$

The value of ε, on the above figures, is 0.05, and this now is a pure number. It has no units and so we can assess it on its merits. It is the ratio of the viscous drag force

[3] These same equations were discussed in connection with the sport of long-jumping) by M. N. Brearley in *Function*, Vol. 3, Part 3.

[4] For an account of the details and for numerical data, see *Basic Mechanics of Fluids* by H. Rouse and J. W. Howe (New York: Wiley, 1953), p. 181.

[5] For a fuller account of the very powerful techniques being employed here, see *Function*, Vol. 10, Part 1.

on the arrow to the gravitational force acting to accelerate the arrow downwards. In our special units this force is 1, and so the viscous force is 20 times less.

This much is theory, but it will turn out not to be so very good a theory for reasons that will be revealed.

For the moment, note that Equations (9) aer much more complicated that Equations (1). Indeed, it is not possible, except in very small cases, to solve them exactly. However, when ε is very small, it is possible to show that the effect of all the extra items is also very small. We may also (even when ε is not very small) solve the equations numerically on a computer. This is shown in Figure 2, where we have put ε=0.05 and used a numerical technique. You will see that the effect of air-resistance is barely noticeable. The range has been reduced from 1 (in our special units) to a figure that works out to be a little over 0.96.

But the range of arrows has been tested and they are found to fall much shorter than this of their theoretical mark. In fact they travel only about 70% of the distance that other experiments say they should.[6] So what has gone wrong?

To see this, consider a very special case, the one in which the arrow is fired vertically. This case is actually one that can be solved exactly. We have, throughout the motion, *x*=0 and so only the *y* equation need concern us. For the *upward* travel this reduces to[7]

$$\ddot{y} = -1 - \varepsilon\dot{y}^2 \tag{10}$$

This equation may be integrated to give

$$\dot{y}^2 = (1 + \frac{1}{\varepsilon})e^{-2\varepsilon y} - \frac{1}{\varepsilon}, \tag{11}$$

and as $\dot{y}$=0 when *y*=*H*, we have

$$H = \frac{1}{2\varepsilon}\ln(1 + \varepsilon) \tag{12}$$

Now, as the arrow goes *up*, its three components (head, shaft and fletchings) all travel along the same path and so the arrow is well and truly streamlined. When it reaches the top. we have our value of *H*, and so in a sense we lose interest in what happens next. However, it is instructive to think, in a general way, about what does occur. The arrow must now come back to earth, and given a reasonable distance of travel, it will eventually return head downwards. In other words it turns around in a tumbling motion; and while this is going on, the arrow will be anything but streamlined! The value of ε will be greatly increased.

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[6] See *New Scientist*, 5 June 1993, pp. 24-25.

[7] on the way *down*, we must write $\ddot{y} = -1 + \varepsilon\dot{y}^2$. Can you see why?

Now these same considerations apply to the more general case as well, although the "tumbling" is distributed more evenly over the trajectory as a whole. The *effective* value of ε will greatly exceed the theoretical value, and indeed the value of ε will alter as the motion proceeds. However, if we take an average value and solve Equations (9) numerically, we find the observed reduction in range for a value of ε of about 0.6. See Figure 3. However, the value of H obtained from Equation (12) should use the value 0.05 derived earlier.

From Equation (12) and the value $\varepsilon=0.05$, we find $H=0.49$, whereas Equation (5) (in our special units, remember) gives 0.5. For R, however, we have the value 0.7. This actually strengthens the argument against the military bow. For tge height reached is no longer merely half the value of the range, but rather 0.49/0.7, i.e. 0.7, and so now (going back to SI units) we would have Vergil claiming that the trees were well over 100m high! This is an even greater unlikelihood than the one we discussed earlier, and so strengthens the case for the interpretation given here.

Further Reading

Equations (9) have been widely studied, but not always in an accessible form. During World War I, the mathematician J. E. :Littlewood[8] was assigned by the British military to study these equations, which are important in gunnery. A few of his results appeared in his book *A Mathematician's Miscellany* and other bits and pieces were published elsewhere. In 1971, *Function*'s British counterpart, *Mathematical Spectrum*, printed a two-part account of his work. (It is rather unlikely that very many of *Spectrum*'s target audience actually read or followed Littlewood's account, which is very heavy going. However, the editors must have published it for its historical importance.) A more accessible popular article is the summary "Ballistics and Projectiles" in *Companion Encyclopedia of the History and Philosopy of the Mathematical Sciences*; see Vol. 2, p. 1069. This gives a lot of the history and many references, but is rather terse when it comes to technical detail.

A problem that sems to be unsolved for the present is that the air-resisted trajectories of Figures 2 and 3 (and very many other cases besides) can be approximated to wonderfully high orders of accuracy by cubic curves. We are unaware of any theoretical reason why this should be so.

Figures

Figure 1. [graph] The standard parabolic trajectory; the path is followed by an arrow in a vacuum. The case drawn is for $\alpha=60°$.

Figure 2. [graph] The path of an arrow in a idealised case but with air resistance. The parameter values are $\varepsilon=0.05$ and $\alpha=45°$. The points shown are those calculated by the numerical programme solving the equations; the curve is interpolated between these.

[8] Who appeared briefly in *Function*, Vol. 19, Part 3, pp. 83-85.

Distance

Figure 3. [graph] A more realistic path. The parameter values for this case are ε=0.06 and α=46°. Again the points are calculated first and then the curve interpolated.